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The Gregory–Laflamme instability and non-uniform generalizations of NUT strings ☆

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ABSTRACT

We explore via linearized perturbation theory the Gregory–Laflamme instability of the NUT string (*i.e.* the $D = 4$ Lorentzian NUT solution uplifted to five dimensions). Our results indicate that the Gregory–Laflamme instability persists in the presence of a NUT charge n, the critical length of the extra-dimension increasing with n for the same value of mass. The non-uniform branch of NUT strings is numerically extended into the full nonlinear regime.

1. Introduction

It is well known that any solution of vacuum general relativity in D-spacetime dimensions can be promoted to a solution of the same theory in $(D+p)$-spacetime dimensions by adding a number of p flat directions. These extra-dimensions are usually supposed to be compact, the model being described by a Kaluza–Klein theory. The simplest such configuration is found by trivially extending to $(D+1)$-spacetime dimensions the Schwarzschild black hole in D-dimensions. The resulting solution corresponds to a uniform black string, the extra-dimension being periodic with an (arbitrary) length L.

Rather unexpectedly, twenty years ago Gregory and Laflamme (GL) made the discovery that, for a given value of L, the Schwarzschild black string is classically unstable against linearized gravitational perturbations below a critical value of the mass [1]. Following this discovery, a branch of non-uniform black string (NUBS) solutions breaking the translational invariance along the periodic direction was found perturbatively from the critical GL string [2–4]. This non-uniform branch was subsequently numerically extended into the full nonlinear regime in [3,5,6]. Further developments have proven that the GL instability is a generic property of black objects in spacetimes with compact extra-dimensions. This includes also rotating solutions [7], non-vacuum solutions [8–10] and configurations with several extra-dimensions compactified on a torus [11] (see [12–14] for reviews of these aspects).

However, most of the work on the stability and phases of black strings has been performed assuming that the solutions approach at infinity the Minkowski spacetime times a circle. Then it is worth inquiring, what happens if we drop these assumptions? Will the GL instability persist? As proven in [15], this is the case in the presence of a negative cosmological constant, since the anti-de Sitter black strings [16,17] are also unstable for small enough values of the event horizon radius.

In some sense, at least for $D = 4$, the minimal deviation from the asymptotic flatness is to include a “dual” or “magnetic” mass in the theory. As explicitly proven by the famous Taub–NUT solution [18–20], general relativity allows for “gravitational dyon” solutions possessing both ordinary (or “electric”) and “magnetic” mass (the NUT charge). In this case, the metric is not asymptotically flat in the usual sense although it does obey the required fall-off conditions. The Taub–NUT spacetime has a number of unusual properties, becoming renowned for being *“a counter-example to almost anything”* [21] and is unlikely to be of interest as a model for a macroscopic object. Nevertheless, the Euclideanized Taub–NUT solution extremizes the gravitational action functional and might play an important role in the context of quantum gravity [22], providing an analogue of instantons in gauge theories.

Let us mention also that the vacuum Taub–NUT solution has been generalized in different directions, by including matter fields or a cosmological constant [23]. There are also some indications that the NUT charge is an important ingredient in low energy string theory (see *e.g.* [24]). However, the pathological features of the vacuum Taub–NUT solution are generic and affect gravitational solutions with “dual” mass in general [25].

Of interest in this work is the fact that, being Ricci flat, the Lorentzian Taub–NUT solution can be promoted to a solution of the $D = 5$ vacuum Einstein equations. However, due to the presence of a NUT charge, the asymptotics are different from the case of a Schwarzschild black string. Then it is interesting to inquire if the NUT strings are also unstable. The main purpose of this Letter is to answer this question. In addition, since all solutions are found

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to possess a GL unstable mode, we construct numerically the corresponding branch of non-uniform configurations.

The Letter is structured as follows: in the next section we review the basic properties of the $D=4$ Lorentzian signature Taub–NUT solution. The $D=5$ uniform string whose stability is investigated is the product of this four-dimensional solution and a circle. In Section 3 we investigate its stability and show results for the GL mode obtained by numerical calculation. The basic properties of the non-perturbative $D=5$ configurations with a dependence of the extra-dimension are discussed in Section 4. We conclude with Section 5 where the results are compiled.

2. The $D=4$ NUT spacetime

The line element of the $D=4$ Taub–NUT spacetime is usually written as

$$ds^2 = \frac{dr^2}{f(r)} + g(r)\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big) - f(r)(dt + 2n\cos\theta\, d\varphi)^2, \tag{1}$$

where

$$f(r) = 1 - \frac{2(mr+n^2)}{r^2+n^2}, \qquad g(r) = r^2 + n^2. \tag{2}$$

This spacetime has two independent parameters, m and n, corresponding to the “electric” and “magnetic” masses, respectively. The NUT charge n plays a dual role to ordinary mass, in the same way that electric and magnetic charges are dual within Maxwell theory [26]. The Killing symmetries of the solution are still translations and $SO(3)$ rotations. However, the spherical symmetry in a conventional sense is lost when the NUT parameter is nonzero, since the rotations act on the time coordinate as well.

This solution has an outer horizon located at

$$r_H = m + \sqrt{m^2+n^2} > 0. \tag{3}$$

Here $f(r_H)=0$ is only a coordinate singularity where all curvature invariants are finite. A nonsingular extension across this null surface can be found just as at the event horizon of a black hole. However, the line-element (1) possesses also an inner horizon at $r_i = m - \sqrt{m^2+n^2} < 0$. The solution derived by Taub in 1951 is valid in the “inner” region with $f(r)<0$, being interpreted as a cosmological model. The metric valid in the outer region $r \geqslant r_H$ (which is of interest in this work), has been derived independently in 1961 by Newman, Unti and Tamburino, being usually called the NUT spacetime.[1]

One can easily see that, for $n \neq 0$, the metric (1) has a singular symmetry axis (defined by $\theta = 0, \pi$). As discussed in [21], these singularities can be removed by appropriate identifications and changes in the topology of the spacetime manifold, which imply a periodic time coordinate.[2] However, in this work, following Ref. [27], we choose a different interpretation of (1), with $-\infty < t < \infty$ and two physical singularities at $\theta=0$ and $\theta=\pi$, respectively. These singularities are interpreted as two semi-infinite counter-rotating rods. Note that the pathology of closed timelike curves is still present in this case, as proven by the fact that, for any θ, the metric component $g_{\varphi\varphi}$ becomes negative for $r<r_c$, with r_c a solution of the equation $\cos 2\theta = \frac{r_c^2+n^2-4n^2 f(r_c)}{r_c^2+n^2+4n^2 f(r_c)}$ [29,30].

As a result, similar to the case of Gödel's rotating universe[3] [31], the Killing vector $\partial/\partial\varphi$ is timelike in a region around the symmetry axis (which extends to infinity for $\theta \to 0,\pi$), and the spacetime is not globally hyperbolic.

This interpretation of the solution leads, however, to an interesting analogy between the angular momentum of the NUT charged spacetimes in general relativity and that of the spinning solitons of the Georgi–Glashow model. To this aim, following [35], we compute the mass and angular momentum of the NUT solution by employing the quasilocal formalism in conjunction with the boundary counterterm method, which avoids the choice of a reference background. In this approach one supplements the gravity action (which contains the Gibbons–Hawking boundary term [36]) by including suitable boundary counterterms, which are functionals of curvature invariants of the induced metric on the boundary. The usual choice [37] for the boundary counterterm is $I_{ct} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\sqrt{2\mathcal{R}}$, where $\mathcal{R}$ is the Ricci scalar of the induced metric on the boundary h_{ij}. Varying the total action with respect to the boundary metric h_{ab}, we obtain the divergence-free boundary stress-tensor $T_{ab} = \frac{1}{8\pi G}(K_{ab} - h_{ab}K - \Psi(\mathcal{R}_{ab} - \mathcal{R}h_{ab}) - h_{ab}\Box\Psi + \Psi_{;ab})$, where K_{ab} is the extrinsic curvature of the boundary and $\Psi = \sqrt{2/\mathcal{R}}$. Provided the boundary geometry has an isometry generated by a Killing vector ξ^i, a conserved charge $\mathfrak{Q}_\xi = \oint_\Sigma d^2S^i\,\xi^j T_{ij}$ can be associated with a closed surface Σ.

Similar to the case of a Schwarzschild black hole, the boundary of the NUT spacetime is taken at constant r, being sent to infinity in the final relations. A straightforward computation gives $8\pi G T_t^t = 2m/r^2 + O(1/r^3)$, which leads to the usual expression of the “electric” mass (which is the charge associated with the Killing vector $\partial/\partial t$)

$$M = \frac{m}{G} = \frac{1}{G}\frac{r_H^2 - n^2}{2r_H}. \tag{4}$$

Interestingly, a gravitational dyon possesses a nonvanishing angular momentum density,

$$8\pi G T_\varphi^t = \frac{4mn\cos\theta}{r^2} + O\big(1/r^3\big). \tag{5}$$

However, one can easily verify that the total angular momentum J (which is the charge associated with the Killing vector $\partial/\partial\varphi$) vanishes. Noticing that $T_{\varphi t}$ is antisymmetric with respect to a reflection in the equatorial plane, one can say that a NUT spacetime consists of two counter-rotating regions, which agrees with the results in [27].

The same quasilocal approach applied to the Kerr black hole leads to the usual expressions for the conserved charges [35]. However, as discussed in [38–40], one may think of the Kerr metric as possessing also a NUT dipole in addition to the usual “electric” mass. Thus we note that the simplest NUT “dyon” in general relativity does not rotate globally, whereas the angular momentum is nonzero for the Kerr solution, which possesses a vanishing net “magnetic” mass. This reveals an interesting analogy with the spinning dyons and dipoles in the Georgi–Glashow model, featuring an $SU(2)$ gauge field and a Higgs field in the adjoint representation. This model possesses globally regular, particle-like solutions, the BPS monopole [41] and dyon [42] being the best known examples. The existence of a profound connection between the angular momentum and the electric and magnetic charges in this theory has been suggested already in the seminal paper [42]. Indeed,

[1] As discussed by Misner in [20], the NUT spacetime can be joined analytically to the Taub spacetime as a single Taub–NUT spacetime.

[2] This comes essentially from the fact that the nondiagonal part of the metric (1) can be generalized to $g_{\varphi t} = -f(r)(2n\cos\theta + n_0)$, with n_0 an arbitrary constant, see *e.g.* the discussion in [27,28].

[3] This analogy becomes more transparent after noticing that the Gödel-type universe corresponds to the boundary metric of the Taub–NUT solution with a negative cosmological constant [32–34].

as discussed in [43,44] the total angular momentum of the solitons endowed with a net magnetic charge vanishes (despite the fact that their angular momentum density can be nonzero). At the same time, the angular momentum of a spinning magnetic dipole is nonzero [45].

Returning to the properties of NUT charged spacetimes, we notice that their thermodynamical description is still poorly understood (the difficulties result mainly from the absence of a global Cauchy surface). Most of the existing results in the literature were found by using a Euclidean approach.[4] The Euclidean version of (1) is obtained by performing the analytic continuation $n \to iN$, $t \to i\tau$; then the regularity of the metric fixes m as a function of N [22]. However, the relevance of the results found on the Euclidean section for the Lorentzian signature solution is unclear [46]. Nevertheless, one can define a temperature of solutions via the surface gravity associated with the Killing vector $\partial/\partial t$

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_H}, \tag{6}$$

and also an (outer) event horizon area

$$A_H = 4\pi\left(r_H^2 + n^2\right). \tag{7}$$

Let us also mention that, although one can write a Smarr-type formula [47], the issue of the first law of thermodynamics for NUT-charged solutions is unclear.

We close this part by remarking that, different from the case of a Schwarzschild black hole, a negative value of the electric mass m is allowed for the NUT solution. Such configurations have $0 < r_H < n$ and do not possess a Schwarzschild limit. Also, since the line-element (1) is invariant under the transformation $n \to -n$, one can take $n > 0$ without loss of generality.

3. The Gregory–Laflamme instability

The line element (1) can trivially be extended to $D = 5$ by adding a dz^2 term to that metric, the extra-coordinate z possessing a periodicity L. It is natural to expect that the resulting uniform solution becomes unstable at critical values of M, n. To determine these critical values, we follow the same approach as in [2]. The starting point is to consider the following ansatz for the non-uniform generalization of the NUT string[5]

$$\begin{aligned} ds^2 = {} & e^{2B(r,z)}\left(\frac{dr^2}{f(r)} + dz^2\right) + g(r)e^{2C(r,z)}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \\ & - f(r)e^{2A(r,z)}(dt + 2n\cos\theta\, d\varphi)^2, \end{aligned} \tag{8}$$

the uniform limit corresponding to $A = B = C = 0$. The problem is thus characterized by two dimensionless parameters: $\mu_1 = MG/L^2$ and $\mu_2 = n/L$. Here M is the mass of the $D = 5$ solutions, as computed from the relation (17) below. (Note that the uniform solutions have $M = Lm/G$.) The limit $\mu_2 \to 0$ corresponds to the Schwarzschild black string solution in a Kaluza–Klein theory, in which case the GL unstable mode occurs for $\mu_1 \simeq 0.0649519$ [1].

In the next step, we perform an expansion of the functions A, B, C in terms of a small parameter ϵ and consider a Fourier series in the z coordinate. In leading order, we assume:

$$X(r,z) = \epsilon X_1(r)\cos(kz) + O\left(\epsilon^2\right), \tag{9}$$

X denoting generically A, B, C and k being the critical wavenumber corresponding to a static perturbation, $k = 2\pi/L$. This expansion is appropriate for studying perturbations at the wavelength which is marginally stable.

We then substitute the form (8) in the general Einstein equations and expand A, B, C according to (9). The system relevant for addressing the stability problem is found by taking the linear terms in the infinitesimal parameter ϵ. Similarly to the $n = 0$ case [2], the Einstein equation $G_r^z = 0$ allows to eliminate the function B_1 in favor of the other functions and to reduce the problem to a system of two differential equations for A_1 and C_1. These equations read:

$$\begin{aligned} & A_1'' + \frac{1}{2}\left(\frac{3f'}{f} + \frac{2g'}{g}\right)A' + \frac{f'}{f}C_1' + \frac{4n^2}{g^2}(A_1 - 2C_1) \\ & \quad + \frac{8n^2 f}{g(gf' + 2fg')}\left(A_1' + 2C_1' + \frac{f'}{2f}A_1 + \frac{g'}{g}C_1\right) \\ & \quad - k^2\frac{A_1}{f} = 0, \\ & C_1'' + \left(\frac{f'}{f} + \frac{2g'}{g}\right)C_1' + \frac{g'}{2g}A_1' + \frac{4n^2}{g^2}(C_1 - A_1) \\ & \quad - \frac{2(2n^2 f + g)}{g(gf' + 2fg')}\left(4C_1' + 2A_1' - \frac{f'}{f}C_1 + \frac{f'}{f}A_1\right) \\ & \quad - k^2\frac{C_1}{f} = 0, \end{aligned} \tag{10}$$

where a prime denotes d/dr. This eigenvalue problem for the wavenumber $k = 2\pi/L$ was solved numerically with suitable boundary conditions. First, the perturbation has to vanish for $r \to \infty$, *i.e.* $\lim_{r\to\infty} A_1, C_1 = 0$. The solutions of the linearized equations should also be regular at the horizon. This leads to a set of specific relations to be satisfied by $A_1(r_H)$, $C_1(r_H)$ and their derivatives.

To integrate Eqs. (10), we have used the differential equation solver COLSYS which involves a Newton–Raphson method [48]. In practice, we have set $r_H = 1$ without loss of generality and computed the corresponding k for a given value of n.

Our numerical results show that the NUT-charged configurations inherit the GL instability of the purely "electric" Schwarzschild black strings. Interestingly, for a given L, the critical value of the mass decreases as n increases, and becomes zero for $n/L \simeq 0.13021$. The strings with $M < 0$ become also unstable for larger values of μ_2. The numerical results are displayed in Fig. 1, where we exhibit the dimensionless quantity $\mu_1 = MG/L^2$ *vs.* the dimensionless ratio between the NUT parameter and the length of the extra-dimension $\mu_2 = n/L$. For completeness, there we show also the value of k as a function of n, for a fixed value $r_H = 1$ of the horizon radius (the $L(n)$ curve is shown in the inset).

4. The non-uniform solutions

4.1. General relations

As usual, the unstable GL mode signals the existence of a branch of solutions with a nontrivial dependence on the extra-coordinate z. These solutions are constructed numerically by using a similar approach to that employed in [5,7] to construct NUBS with usual Kaluza–Klein asymptotics.

[4] An interesting result here is that the entropy of such solutions generically does not obey the simple "quarter-area law", see the discussion in [33].

[5] Similar to the case of a $D = 4$ NUT spacetime, the singularities at $\theta = 0, \pi$ can be eliminated by a coordinate transformation together with a periodic identification of t.

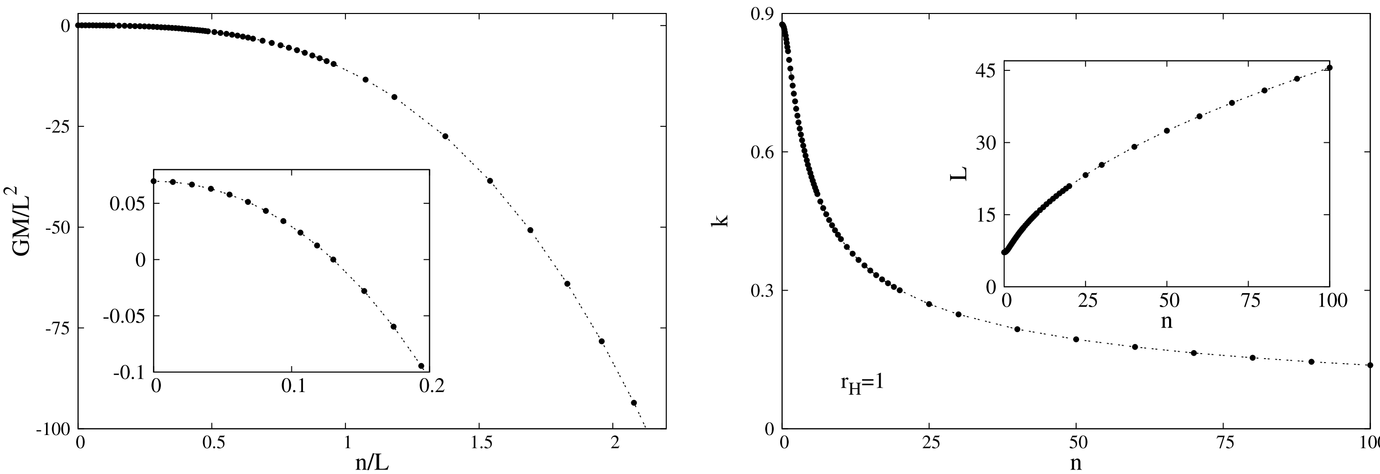

Fig. 1. *Left*: The critical mass M is shown as a function of the NUT parameter n (these quantities are given in units set by the length L of the extra-dimension). *Right*: The wavelength $k = 2\pi/L$ is shown as a function of the NUT parameter n for solutions with a fixed horizon radius $r_H = 1$ (the corresponding $L(n)$ curve is shown in the inset).

The Einstein equations $G_t^t = 0$, $G_r^r + G_z^z = 0$ and $G_\theta^\theta = 0$ yield for the functions A, B, C the set of equations[6]

$$A'' + \frac{\ddot{A}}{f} + A'^2 + \frac{\dot{A}^2}{f} + \frac{2rA'}{g} + \frac{3f'A'}{2f} + \frac{f'C'}{f} + 2\left(A'C' + \frac{\dot{A}\dot{C}}{f}\right) + \frac{f'r}{fg} + \frac{f''}{2f} + \frac{2e^{2A+2B-4C}n^2}{g^2} = 0, \tag{11}$$

$$B'' + \frac{\ddot{B}}{f} - 2\left(A'C' + \frac{\dot{A}\dot{C}}{f}\right) - C'^2 - \frac{\dot{C}^2}{f} - \frac{2rA'}{g} + \frac{f'B'}{2f} - \frac{(2rf + gf')}{fg}C' + \frac{1}{fg}\left(e^{2B-2C} - rf'\right) - \frac{r^2}{g^2} + \frac{e^{2A+2B-4C}n^2}{g^2} = 0, \tag{12}$$

$$C'' + \frac{\ddot{C}}{f} + 2\left(C'^2 + \frac{\dot{C}^2}{f}\right) + A'C' + \frac{\dot{A}\dot{C}}{f} + \frac{rA'}{g} + \frac{f'C'}{f} + \frac{4rC'}{g} + \frac{1}{fg}\left(rf' - e^{2B-2C}\right) + \frac{1}{g} - \frac{2e^{2A+2B-4C}n^2}{g^2} = 0, \tag{13}$$

where a prime denotes $\partial/\partial r$, and a dot $\partial/\partial z$. (Note that the $D = 5$ equations in Ref. [5] are recovered for $n = 0$.)

To solve these equations, we use the same approach as for $n = 0$, and introduce a new radial coordinate $\tilde{r}$, where $r = \sqrt{r_H^2 + \tilde{r}^2}$ (*i.e.* the horizon resides at $\tilde{r} = 0$). Utilizing the reflection symmetry of the solutions w.r.t. $z = L/2$, the solutions are constructed subject to the boundary conditions

$$\partial_z A|_{z=0,L/2} = \partial_z B|_{z=0,L/2} = \partial_z C|_{z=0,L/2} = 0,$$
$$A|_{\tilde{r}=0} - B|_{\tilde{r}=0} = d_0,$$
$$\partial_{\tilde{r}} A|_{\tilde{r}=0} = \partial_{\tilde{r}} C|_{\tilde{r}=0} = 0 \tag{14}$$

(where the constant d_0 is related to the Hawking temperature of the solutions (18)), together with

$$A|_{\tilde{r}=\infty} = B|_{\tilde{r}=\infty} = C|_{\tilde{r}=\infty} = 0, \tag{15}$$

such that the uniform background $NUT \times S^1$ is approached asymptotically. Regularity further requires that the condition $\partial_{\tilde{r}} B|_{\tilde{r}=0} = 0$ holds for the solutions.

The asymptotic form of the relevant metric components is

$$g_{tt} \simeq -1 + \frac{c_t}{r}, \qquad g_{zz} \simeq 1 + \frac{c_z}{r}, \tag{16}$$

and contains two parameters c_t and c_z encoding the global charges of the solutions (with $c_t = 2m$, $c_z = 0$ for uniform configurations).

The global charges of the NUT strings are the mass M and the tension T. In their computation, it is convenient to use again the quasilocal formalism augmented by the counterterm approach.[7] M and T are charges associated with the asymptotic Killing vectors $\partial/\partial t$ and $\partial/\partial z$, respectively, their expressions being analogous to those valid in the $n = 0$ limit, with[8]

$$M = \frac{L}{4G}(2c_t - c_z), \qquad T = \frac{1}{4\pi G}(c_t - 2c_z). \tag{17}$$

Other quantities of interest are the Hawking temperature and the horizon area of the non-uniform solutions

$$T_H = \frac{1}{4\pi r_H} e^{A_0 - B_0}, \qquad A_H = 4\pi L\left(r_H^2 + n^2\right) \int_0^L e^{B_0 + 2C_0}\, dz, \tag{18}$$

where $A_0(z)$, $B_0(z)$, $C_0(z)$ are the values of the metric functions on the event horizon $r = r_H$.

To obtain a measure of the deformation of the solutions, we define the non-uniformity parameter [2]

$$\lambda = \frac{1}{2}\left(\frac{\mathcal{R}_{\max}}{\mathcal{R}_{\min}} - 1\right), \tag{19}$$

where $\mathcal{R}_{\max}$ and $\mathcal{R}_{\min}$ represent the maximum and minimum radii of the two-sphere on the horizon.

We remark also that Eqs. (11)–(13) are left invariant by the transformation $r \to r/p$, $z \to z/p$, $r_H \to r_H/p$, $n \to n/p$, with p a positive integer. Therefore, a new family of vacuum solutions with the same length of the extra-dimension can be generated in this way (see *e.g.* [49] for a detailed discussion of this procedure

[6] Note that the Einstein equations $G_z^r = 0$, $G_r^r - G_z^z = 0$ are not automatically satisfied, yielding two constraints. However, following [3], one can show that these constraints are satisfied as a consequence of the Bianchi identities. Also, one can show that all other Einstein equations are either linear combinations of those used to derive (11)–(13) or are identically zero.

[7] For $D = 5$ solutions, the appropriate expression of the counterterm is $I_{ct} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x \sqrt{-h}\sqrt{2\mathcal{R}}$.

[8] Note that the non-uniform solutions possess also a nonzero angular momentum density. However, similar to the case of the $D = 4$ NUT solution, the total angular momentum vanishes.

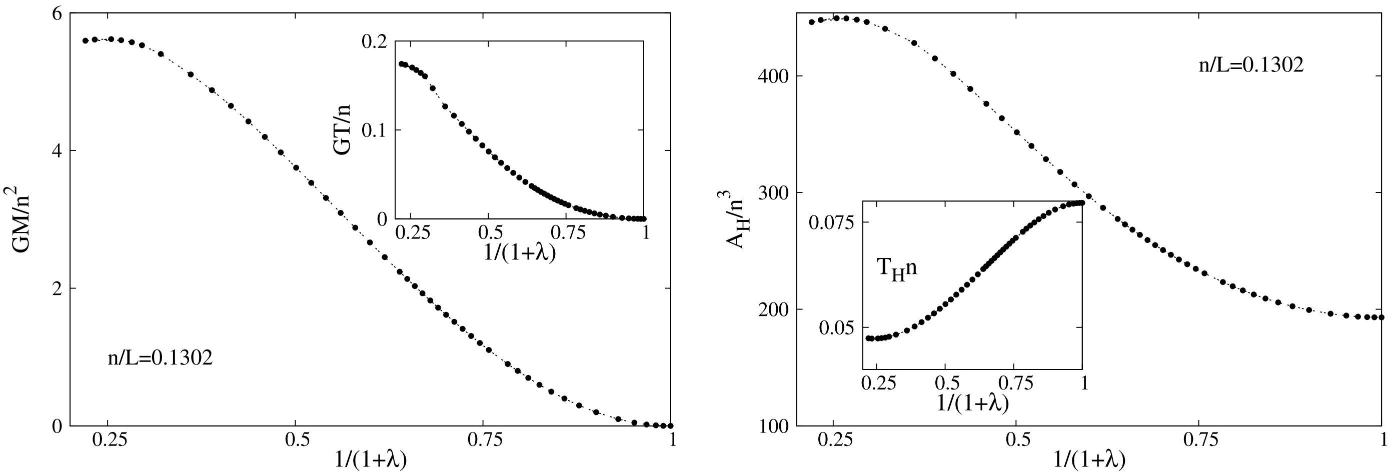

Fig. 2. The mass M, the tension T, the temperature T_H and the area A_H are shown in units of n as functions of the non-uniformity $1/(1+\lambda)$, for a family of non-uniform strings with $\mu_2 = n/L = 0.1302$.

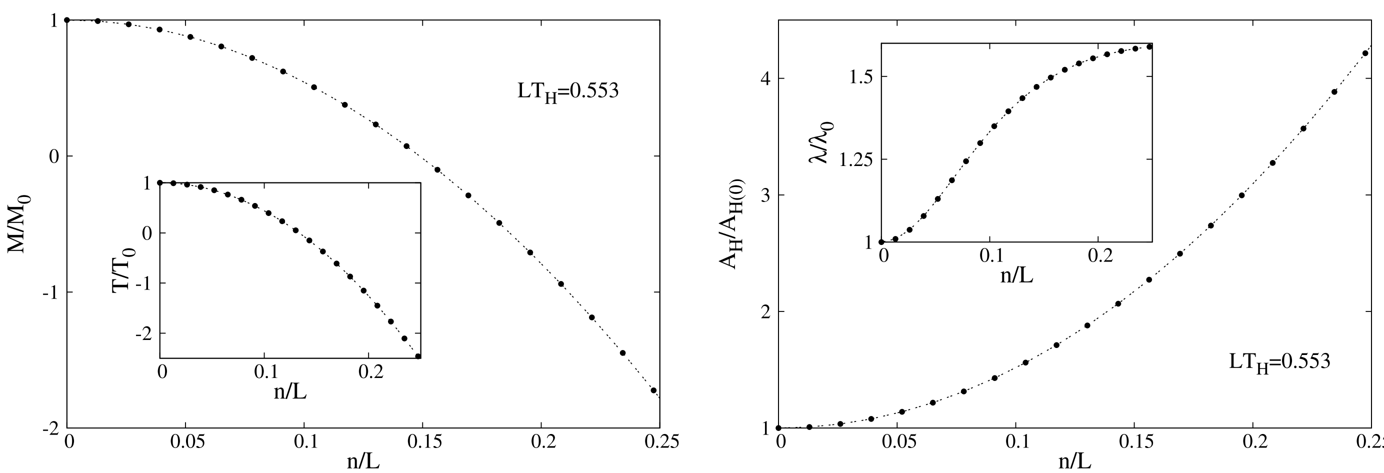

Fig. 3. The mass M, the tension T, the area A_H and the non-uniformity parameter λ of a family of solutions with $LT_H = 0.553$ are shown in units of the corresponding $n = 0$ NUBS solution (denoted by M_0, T_0, $A_{H(0)}$ and λ_0), as functions of the dimensionless ratio n/L.

for the $n = 0$ case). Their relevant quantities, expressed in terms of the initial solution, read $M^{(p)} = M/p$, $T^{(p)} = T/p$, $T_H^{(p)} = pT_H$, $A^{(p)} = A_H/p^2$.

4.2. The solutions

The nonlinear elliptic partial differential equations (11)–(13) are solved numerically, subject to the boundary conditions (14), (15). The numerical calculations are based on the Newton–Raphson method and are performed with help of the professional package FIDISOL/CADSOL [50], which provides also an error estimate for each unknown function. (See Ref. [5] for a detailed description of the numerical scheme.)

The input parameters of the problem are the horizon coordinate r_H, the temperature T_H, the NUT charge n and the asymptotic length L of the compact z-direction.

For a given n, a branch of non-uniform solutions is obtained by starting at the critical point of the uniform configurations and varying the boundary parameter d_0, which enters Eq. (14), relating the values of the functions A and B at the horizon. Our numerical results show that the properties of the solutions are rather similar to the case of the non-uniform generalizations of the $n = 0$ Schwarzschild black string. In particular, the functions A, B, C have a similar shape to that displayed in Ref. [5], exhibiting extrema at $z = 0$ at the horizon. As λ increases, the extrema increase in height and become increasingly sharp.

Some numerical results are displayed in Fig. 2, where we exhibit the mass M, the tension T, the temperature T_H and the horizon area A_H versus the parameter λ for a family of non-uniform solutions with $\mu_2 = n/L = 0.1302$. In that plot, M, T, T_H and A_H are given in units of n, with $\lambda = 0$ corresponding to the uniform solution. One can see that the mass and horizon area assume a maximal value for a value of $\lambda = \lambda_{ex}$, where the temperature assumes a minimal value.

Non-uniform strings can also be obtained by starting from $n = 0$ NUBSs with a given temperature (as specified by the parameter d_0) and length of the extra-dimension, and then slowly increasing the value of the NUT charge. The numerical results suggest that any NUBS possess generalizations with $n \neq 0$. No upper bound on n appears to exist, although the numerical integration becomes more difficult with increasing n.

In Fig. 3 we exhibit the mass M, tension T, horizon area A_H and non-uniformity parameter λ for non-uniform string solutions with $LT_H = 0.553$, in units of the corresponding $n = 0$ solution, versus the parameter $\mu_2 = n/L$. As one can see, both the mass and tension decrease with n, becoming negative for large enough values of the NUT charge. At the same time, the horizon area and the non-uniformity parameter increase.

5. Further remarks

The main purpose of this work was to investigate the stability of the Lorentzian NUT solution uplifted to $D = 5$ dimensions. Even if one's primary interest is in solutions with usual Kaluza–Klein asymptotics, we hope that, by widening the context to solutions with NUT charge, one may achieve a deeper appreciation of the theory. In particular, one may hope to determine more general features of the string solutions, independent of whether or not they contain a "magnetic" mass. Expanding around the uniform solution and solving the eigenvalue problem numerically, our results indicate that the GL instability persists for $n \neq 0$ configurations. Moreover, for a given length of the extra-dimension, a $D = 5$ NUT-charged solution becomes unstable for a smaller value of the mass as compared to the Schwarzschild black string.

We also constructed numerically the corresponding non-uniform strings emerging from the branch of marginally stable uniform solutions. The properties of these solutions are rather similar to the well-known $n = 0$ case. An interesting point which remains to be clarified is the phase diagram of the $D = 5$ solutions approaching at infinity a $NUT \times S^1$ background. For $n = 0$, apart from the black string solutions, the Kaluza–Klein theory possesses also a branch of black hole solutions with an S^3 topology of the event horizon. There is now convincing evidence that the non-uniform string branch and the black hole branch merge at a topology changing solution. Based on the numerical results in Section 4, we expect that a similar picture should be valid also for the configurations in this work. The conjectured horizon topology changing transition should be approached again for $\lambda \to \infty$. However, the construction of the $D = 5$ nutty solutions with an S^3 horizon topology still represents a numerical challenge.

Finally, let us remark that the NUT solution (1) possesses higher dimensional $D = 2K + 2$ generalizations (see *e.g.* [51] and the references there). Their main properties (in particular the presence of closed timelike curves) follow closely the four-dimensional case. These generalized NUT solutions can also be uplifted to $D = 2K + 3$ dimensions and are likely to possess as well a GL unstable mode.

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